Overcoming the $su(2^n)$ sufficient condition for the coherent control of n-qubit systems

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Abstract

We study quantum systems with even numbers N of levels that are completely state-controlled by unitary transformations generated by Lie algebras isomorphic to $sp(N)$ of dimension $N(N+1)/2$. These Lie algebras are smaller than the respective $su(N)$ with dimension N^2-1. We show that this reduction constrains the Hamiltonian to have symmetric energy levels. An example of such a system is an n-qubit system. Using a geometric representation for the quantum wave function of a finite system, we present an explicit example that shows a two-qubit system can be controlled by the elements of the Lie algebra $sp(4)$ (isomorphic to $spin(5)$ and $so(5)$) with dimension ten rather than $su(4)$ with dimension fifteen. These results enable one to envision more efficient algorithms for the design of fields for quantum-state engineering, and they provide more insight into the fundamental structure of quantum control.

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I. INTRODUCTION

The coherent control of an N-level quantum system is of interest in fields such as chemical dynamics [1], quantum information processing [2], and quantum communication [3]. It is well known [4, 5] that for an N-level system to be completely controllable, it is sufficient that the free-evolution Hamiltonian, along with the interaction Hamiltonian (which could involve a sequence of steps) and all possible commutators among them, form a Lie algebra of dimension N^2, which in general is taken to be $u(N)$. Recently, it has been shown [6] that state-to-state controllability can be achieved with a Lie algebra isomorphic to $sp(N)$ with dimension $N(N+1)/2$. (We use the notation $sp\,(N)$ for the algebra of the group $Sp\,(N)$ of $N \times N$ unitary symplectic matrices, as for example in the text by Jones [7]. Other authors denote the same group by $Usp(N)$ [8] or $Sp(N/2)$ [6].) In this paper, we show by calculating the Cartan subalgebra that this reduction places a restriction on the types of systems that can be state-to-state controlled. Specifically, not only do the systems have to have an even number of energy levels [6], their field-free energy levels must be symmetrically distributed about an average. An example of such a system is a multi-qubit system that has $N = 2^n$ energy levels. This result in quantum control is important both for developing optimal control schemes in quantum computing [9, 10] and for finding algorithms to calculate applied fields for quantum-state engineering [11].

The control equations can be derived from the time-dependent Schrödinger equation

$$\dot{x}(t) = \left(A + \sum_{i=1}^{m} u_i(t) B_i \right) x(t), \tag{1}$$

where the state vectors $x(t) \in \mathbb{C}^n$, give the amplitudes in a basis of free-evolution eigenstates, A and B_i are constant matrices, and the real scalar functions $u_i(t)$ are the control fields. The evolution of an N-level system can be studied by integrating the corresponding matrix equation in which $x(t)$ is replaced by a matrix $X(t)$, each column of which represents an independent state; one follows the evolution of $X(t)$ from the identity matrix $X(0) = I$. If A and B_i are anti-Hermitian, the solutions of $x(t)$ have constant norms $|x(t)|$ and can thus be viewed as lying on a sphere, and the groups that define the complete controllability of Eq. (1) for general systems are those summarized in [4].

In this paper, we study and independently demonstrate a sufficient condition suggested by Refs. [4, 6, 12] for establishing controllability of a common class of systems that uses

$sp(N)$ Lie algebras, which are smaller, namely of dimension $N(N+1)/2$, compared with N^2-1 for $su(N)$ or N^2 for $u(N)$. We show that the Cartan subalgebra of $sp\,(N)$ restricts its application to systems where the free-evolution Hamiltonian has a symmetric distribution of energy levels about an average. These systems are a subset of the general ones discussed in references [4, 5]. As an example, we illustrate explicitly that a system with four levels (a two-qubit system) is controllable with $sp(4)$, which is isomorphic to the $spin(5)$ and $so(5)$ algebras, and which has 10 dimensions and is thus smaller than $su(4)$ with its 15 dimensions. Similarly, a system with eight levels (a three-qubit system) is controllable with a Lie algebra of dimension 36, significantly smaller than $su(8)$ with its 63 dimensions.

II. SUFFICIENT CONDITION FOR STATE CONTROLLABILITY

The wave function Ψ is constructed as a unitary transformation of a reference or pass state [13] represented in a geometric representation by the primitive projector P. The unitary transformation is an exponential operator of anti-Hermitian elements of the Lie algebra for the system:

$$\Psi = e^{\mathbf{a}}P,\ \mathbf{a} \in \text{Lie algebra}, \tag{2}$$

and P can be represented by the singular matrix

$$P = \begin{pmatrix} 1 & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

with 0s everywhere except at the upper-left diagonal position. One can verify the normalization $\mathrm{tr}(\Psi^\dagger\Psi) = \mathrm{tr}(Pe^{-a}e^{a}P) = \mathrm{tr}(P) = 1$. In this form, the wave function, as an element of a Clifford algebra, represents an arbitrary single state of the system as a square matrix, corresponding to X mentioned in the previous paragraph but with a single nonvanishing column on the leftmost side.

Our sufficient condition for a Lie algebra that governs the pure-state control of a quantum system is based on the following: the parametrization of the wave function using unitary exponential operators $e^{\mathbf{a}}$ of the Lie algebra defines a complete control scheme if we are able to reach an arbitrary ray in the complete state space. We illustrate the procedure first in general terms and then give explicit examples.

We require that for any pair of basis states ψ_j, ψ_k of the state space, there exists an anticommuting pair of antihermitian elements $\mathbf{a}_{kj}, \mathbf{b}_{kj}$ of the algebra that relates them:

$$\begin{aligned} &\psi_k = \mathbf{a}_{kj}\psi_j = -i\mathbf{b}_{kj}\psi_j \qquad (3) \\ &\mathbf{a}_{kj} = -\mathbf{a}_{kj}^\dagger, \ \mathbf{b}_{kj} = -\mathbf{b}_{kj}^\dagger, \ \mathbf{a}_{kj}\mathbf{b}_{kj} + \mathbf{b}_{kj}\mathbf{a}_{kj} = 0. \end{aligned}$$

It is important to remember here that the basis states have the projective form (2) of a minimal left ideal of the algebra. Assuming unit normalization $(\mathbf{a}_{kj})^2 = -1 = (\mathbf{b}_{kj})^2$, it follows that we can write $\psi_k = \exp[\mathbf{a}_{kj}\pi/2]\,\psi_j = -i\exp[\mathbf{b}_{kj}\pi/2]\,\psi_j$, and the more general superposition,

$$\begin{aligned} &\exp\left[(\mathbf{a}_{kj}\cos\phi + \mathbf{b}_{kj}\sin\phi)\frac{\theta}{2}\right]\psi_j = \psi_j\cos\frac{\theta}{2} + \psi_k e^{i\phi}\sin\frac{\theta}{2} \\ &= \exp\left(\mathbf{c}_{kj}\frac{\phi}{2}\right)\exp\left(\mathbf{a}_{kj}\frac{\theta}{2}\right)\exp\left(-\mathbf{c}_{kj}\frac{\phi}{2}\right), \qquad (4) \end{aligned}$$

is expressed as a continuous "rotation" with real angle parameters θ, ϕ, in state space, where

$$\mathbf{c}_{kj} = \frac{1}{2}\left[\mathbf{a}_{kj}\mathbf{b}_{kj} - \mathbf{b}_{kj}\mathbf{a}_{kj}\right]$$

is another element of the Lie algebra and we noted that $\mathbf{c}_{kj}\mathbf{a}_{kj} = \mathbf{b}_{kj}$. Products of such unitary operators allow transitions from one basis state to any linear combination of the states. One additional element $\mathbf{b}_{jj}$ is needed to simply change the complex phase of ψ_j :

$$i\psi_j = \mathbf{b}_{jj}\psi_j. \qquad (5)$$

The elements $\mathbf{a}_{kj}, \mathbf{b}_{kj}, \mathbf{c}_{kj}$ are generators of the control group and represent the effect of coupling fields. Given any initial basis state ψ_j, a general state of the system is a real linear combination

$$\Psi = \sum_k \left(\alpha_{kj}\mathbf{a}_{kj} + \beta_k j\mathbf{b}_{kj}\right)\psi_j, \ \ \alpha_{kj}, \beta_{kj} \in \mathbb{R} \qquad (6)$$

of the $\mathbf{a}_{kj}$ and $\mathbf{b}_{kj}$ generators operating on ψ_j, where for notational convenience we write $\mathbf{a}_{jj} = 1$. In practice, the elements $\mathbf{a}_{kj}, \mathbf{b}_{kj}, \mathbf{c}_{kj}$ are members of the same small set. As we demonstrate below, a set of N distinct elements is sufficient to generate a Lie algebra of $N(N+1)/2$ dimensions.

Calculating the Lie algebra of a higher-dimensional system can require intensive computations, but there is an elegant and efficient approach using techniques of Clifford's geometric

algebra. The N-level quantum system can be described using multivectors in a geometric algebra. The bivectors are well known as generators of the spin groups, and it has been shown [14] that in fact every Lie group can be represented as a spin group. Here we introduce the possibility of using the full set of anti-Hermitian multivectors (including, for example, trivectors and six-vectors) to generate the control group. We illustrate our method with examples of one- and two-qubit systems, and then generalize to show how the control of an n-qubit system can be achieved by a Lie algebra generally smaller than $su(N)$.

A. Example: Single Qubit Control

In the simplest example, Clifford's geometric algebra $C\ell_3$ of three-dimensional Euclidean space enables us to describe a single qubit ($N = 2$) [15]. In this case, Pauli spin matrices can represent the three orthonormal vectors (basis elements of grade 1): $\mathbf{e}_j = \boldsymbol{\sigma}_j,\ j = 1, 2, 3$. The products of grade 2

$$\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2,\ \mathbf{e}_{23} = \mathbf{e}_2\mathbf{e}_3,\ \mathbf{e}_{31} = \mathbf{e}_3\mathbf{e}_1\ , \tag{7}$$

form a basis for the bivector space and generate rotations. There is a single independent element of grade 3, namely the trivector

$$\mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\ , \tag{8}$$

whose matrix representation is i times the unit matrix. These elements along with the identity span the full linear space of the closed algebra $C\ell_3$.

We can take the basis states of the system to be $\psi_0 = P$ and $\psi_1 = \mathbf{e}_{13}P$. Then we note by the "pacwoman" property of projectors, [15, 16] namely $\mathbf{e}_3 P = P$, that $i\psi_1 = i\mathbf{e}_1 P = \mathbf{e}_{23}P$ and $i\psi_0 = \mathbf{e}_{12}P$. The $N = 2$ generators $\mathbf{a}_{10} = \mathbf{e}_{13},\ \mathbf{c}_{10} = -\mathbf{e}_{12}$, generate the control Lie algebra $spin(3)$, which is isomorphic to $su(2)$, $so(3)$, and $sp(2)$. An arbitrary state can be expressed by

$$\Psi = \exp\left(-\mathbf{e}_{12}\frac{\phi}{2}\right)\exp\left(\mathbf{e}_{13}\frac{\theta}{2}\right)\exp\left(-\mathbf{e}_{12}\frac{\chi}{2}\right)P,$$

which, in fact, is just the Euler-angle expression for the Bloch-sphere representation the state [15]. Note that since the exponents form a closed Lie algebra, no generators outside of the algebra arise from an expansion of the unitary operator [11].

4D and 5D	7D	
$\mathbf{e}_1 = \ \boldsymbol{\sigma}_3 \otimes \boldsymbol{\sigma}_1$	$\mathbf{e}_1 = 1 \otimes \boldsymbol{\sigma}_3 \otimes \boldsymbol{\sigma}_1$	$\mathbf{e}_6 = \boldsymbol{\sigma}_1 \otimes \boldsymbol{\sigma}_1 \otimes 1$
$\mathbf{e}_2 = \ \boldsymbol{\sigma}_3 \otimes \boldsymbol{\sigma}_2$	$\mathbf{e}_2 = 1 \otimes \boldsymbol{\sigma}_3 \otimes \boldsymbol{\sigma}_2$	$\mathbf{e}_7 = \boldsymbol{\sigma}_2 \otimes \boldsymbol{\sigma}_1 \otimes 1$
$\mathbf{e}_3 = \ \boldsymbol{\sigma}_3 \otimes \boldsymbol{\sigma}_3$	$\mathbf{e}_3 = 1 \otimes \boldsymbol{\sigma}_3 \otimes \boldsymbol{\sigma}_3$	
$\mathbf{e}_4 = -\boldsymbol{\sigma}_2 \otimes 1$	$\mathbf{e}_4 = \ 1 \otimes \boldsymbol{\sigma}_2 \otimes 1$	
$\mathbf{e}_5 = -\boldsymbol{\sigma}_1 \otimes 1$	$\mathbf{e}_5 = \boldsymbol{\sigma}_3 \otimes \boldsymbol{\sigma}_1 \otimes 1$	

TABLE I: A matrix representation of orthonormal vectors for some dimensions. The 4×4 matrix representation for $5D$ is not faithful for the universal Clifford algebra $C\ell_5$ (it is a homomorphism rather than an isomorphism) but does represent all bivectors uniquely and is therefore adequate for state control.

We assume a basis for the system in which the free-evolution Hamiltonian H_0 is diagonal. Since commutators (Lie products) of H_0 with the control transformations are to remain within the Lie algebra, we need to construct H_0 from the unit matrix plus elements of the Lie algebra. To ensure that H_0 is diagonal, its contributions from the Lie algebra are restricted to the Cartan subalgebra, defined as the largest set of commuting generators of the Lie algebra. For the two-state system, the Cartan subalgebra of $su\,(2)$ comprises a single element, namely the generator $\mathbf{e}_{12} = \sigma_1\sigma_2 = i\sigma_3$. We thus construct a general Hamiltonian for a two-level system (apart from an offset energy proportional to the unit matrix) as

$$H_0 = -i\mathbf{e}_{12}\omega = \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

III. GEOMETRIC REPRESENTATION OF MULTI-QUBIT CONTROL

For systems of multiple qubits, the orthogonal unit vectors of the appropriate Clifford algebra can be represented as tensor products (Kronecker products) of the Pauli matrices as shown in Table I. Bivectors, trivectors, etc., can be obtained by the product of the unit orthogonal vectors among themselves.

Any homogeneous multivector (comprising elements of a single grade g) in the real Clifford algebra $C\ell_n$ for an n-dimensional Euclidean space can be classified as Hermitian or anti-

Hermitian according to its grade. Elements of grade 0,1,4,5,8,9 or generally whenever the grade is 0 or 1 *mod* 4, are Hermitian whereas those of other grades are anti-Hermitian. This is important because the bivectors on one hand, as well as the complete set of anti-Hermitian multivectors on the other hand, form Lie algebras of compact groups. In some algebras for Euclidean spaces of odd dimension, as for example in $C\ell_3$ or $C\ell_7$, the highest-grade multivector (the volume element) is anti-Hermitian but commutes with every element of the algebra, and it therefore must be excluded from the set of all anti-Hermitian elements that generates the Lie algebra.

A. Example: Two-Qubit Control

The four-level system, understood as comprising two qubits, is controlled using the bivectors plus trivectors of the Clifford algebra $C\ell_4$ of four-dimensional Euclidean space or, equivalently, by the bivectors of a Clifford algebra for a five-dimensional Euclidean space, namely the nonuniversal Clifford algebra $C\ell_5\,(1+\mathbf{e}_{12345})\,/2$, a left ideal of $C\ell_5$, which is isomorphic to $C\ell_4$. These bivectors generate the $spin(5)$ algebra, which is isomorphic to $so\,(5)$ and to $sp(4)$. The projector for two-qubits, can be represented in terms of bivectors $\mathbf{e}_{jk}$ (see Table I) by

$$P = \frac{1}{4}(1 - i\mathbf{e}_{12})(1 + i\mathbf{e}_{45}). \tag{10}$$

The dimension of the control algebra is ten. Because the elements form a closed algebra, in this case $spin\,(5)$, we know that no other generators are needed for state control. The Cartan subalgebra in this case is two dimensional so that there are two diagonal generators among the $spin(5)$ generators, from which we can construct the free-evolution Hamiltonian (apart from a constant offset and with $\hbar = 1$)

$$H_0 = \frac{i}{2}(\omega_2 + \omega_1)\mathbf{e}_{45} - \frac{i}{2}(\omega_2 - \omega_1)\mathbf{e}_{12}\ . \tag{11}$$

This Hamiltonian has symmetric eigenenergies as represented in figure (2)

$$H_0 = \begin{pmatrix} \omega_2 & 0 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 \\ 0 & 0 & -\omega_1 & 0 \\ 0 & 0 & 0 & -\omega_2 \end{pmatrix}. \tag{12}$$

a	c	Transitions
0	$\mathbf{e}_{12}$	$0 \leftrightarrow 0, 1 \leftrightarrow 1, 2 \leftrightarrow 2, 3 \leftrightarrow 3,$
$\mathbf{e}_{13}$	$\mathbf{e}_{12}$	$0 \leftrightarrow 1, 2 \leftrightarrow 3$
$\mathbf{e}_{24}$	$\mathbf{e}_{12}$	$1 \leftrightarrow 2, 0 \leftrightarrow 3$
$\mathbf{e}_{35}$	$\mathbf{e}_{45}$	$0 \leftrightarrow 2, 1 \leftrightarrow 3$

TABLE II: Generators for transition operators in 2-qubit systems (see text).

The sufficient condition for state controllability thus leads to a class of systems *with energy levels symmetrically distributed about a center,* such as those that can be found in trapped-ion qubits [17] or coupled spins [18].

The unitary transition operators among the eigenstates can be expressed [see Eq. (4)] in the form $\exp(\mathbf{c}\phi/2)\exp(\mathbf{a}\theta/2)\exp(-\mathbf{c}\phi/2)$, where θ determines the magnitudes of the state amplitudes and ϕ gives the relative phase. The transition between states is complete when $\theta = \pi$, as in a π pulse. Table II shows the generators $\mathbf{a}, \mathbf{c}$ for each transition in the 2-qubit system. Note that with $\theta = \pm\pi/2$, the partial transitions $1 \leftrightarrow 2, 0 \leftrightarrow 3$ induced by the coupled-qubit bivector $\mathbf{e}_{24}$, create the four entangled Bell states.

Thus all the transitions, together with control of the relative phase, require no more than the five nonzero elements in Table II and commutators of these elements give all ten independent elements of *spin* (5). However, only four of the five are required in a minimal set, since for example $\mathbf{e}_{45}$ can be obtained from the other four:

$$\begin{aligned}
&\frac{1}{2}[\mathbf{e}_{12}, \mathbf{e}_{24}] = \mathbf{e}_{14} \\
&\frac{1}{2}[\mathbf{e}_{13}, \mathbf{e}_{35}] = \mathbf{e}_{15} \\
&\frac{1}{2}[\mathbf{e}_{15}, \mathbf{e}_{14}] = \mathbf{e}_{45} = \exp(\mathbf{e}_{15}\pi/4)\,\mathbf{e}_{14}\exp(-\mathbf{e}_{15}\pi/4)\,.
\end{aligned}$$

Fewer than four is easily seen to be insufficient to generate all the elements of *spin* (5), so that four is the number of elements that is necessary and sufficient for state control of an arbitrary 2-qubit system. The anti-Hermitian multivectors used to define controllable schemes are summarized in Table III for small systems.

Clifford Algebra	qubits	N	Lie algebra	Dim
$C\ell_3$ Bivectors only	1	2	$su(2)$	3
$C\ell_4$ Anti-Hermitian	2	4	$sp(4)$	10
$C\ell_5$ Bivectors only	2	4	$spin(5) \cong sp\,(4)$	10
$C\ell_6$ Anti-Hermitian	3	8	$sp(8)$	36

TABLE III: Lie algebras and their controllable n-qubit systems. $N = 2^n$ is the number of levels and also the minimum number of elements needed to generate the entire algebra.

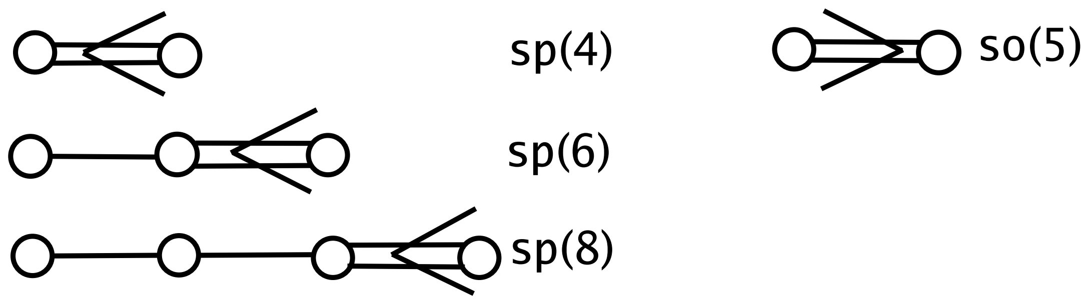

FIG. 1: Dynkin diagrams corresponding to some of the Symplectic Lie algebras in low dimensions and the case for $so(5)$, which is isomorphic to $spin\,(5)$.

The Lie algebras of interest are of dimension $N(N+1)/2$, which is the same as the dimension of the symplectic Lie algebras $sp(N)$ (for even N). The Dynkin diagrams for lower dimension are shown in figure 1 including the case of $sp(4)$ to show the isomorphism with $so(5)$ and thus with $spin\,(5)$.

IV. EXPLICIT CONTROL SCHEME

The method is readily extended to higher even values of N. An explicit control scheme can show that an arbitrary superposition in a quantum system with even number of energy levels that are symmetrically distributed about an offset can be produced from another arbitrary superposition using a set of fields, and the Lie algebras generated by these field-couplings are of dimension $N(N+1)/2$. This scheme is based on the subspace controllability theorem [19] that describes the method of transferring any superposition of states to any other superposition through a pivot state (pass state). This builds on the work done by

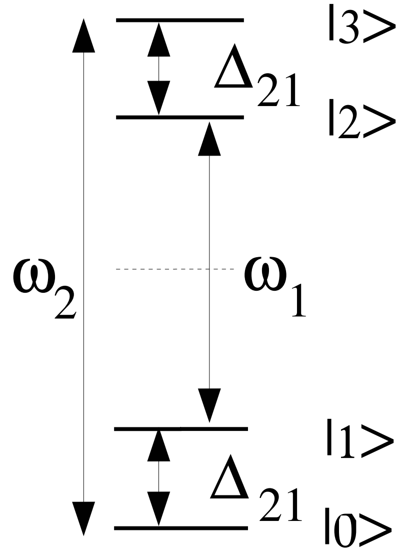

FIG. 2: Symmetric energy levels of a 2-qubit system with two interaction fields.

Eberly and coworkers on the control of harmonic oscillator states [20, 21].

In the general case of the even N-level system with symmetric energies, this scheme is implemented by transferring population in any superposition of states to the ground state $|0\rangle$ through a sequential application of fields. (In Table II, we show the fields connecting all energy states, and in practise some of these these may correspond to qubit-qubit couplings. However, this scheme will succeed with any sequentially connected quantum transfer graph [22]) To obtain any arbitrary final-state superposition, the time-reversed sequence of fields is applied starting from $|0\rangle$. Since the system is finite, we conclude that it is arbitrarily controllable. Note that n-qubit systems are all cases of the general even-level system with symmetric energy distributions.

The control algebra for this scheme contains only $N(N+1)/2$ elements, which can be always constructed defining an initial set of N generators with representation matrices of

the form

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & & -1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \dots$$

$$\begin{pmatrix} 0 & i & 0 & \dots & 0 & 0 & 0 \\ i & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & i \\ 0 & 0 & 0 & \dots & 0 & i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & i & & 0 & 0 & 0 \\ 0 & i & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & i & 0 \\ 0 & 0 & 0 & & i & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \dots . \tag{13}$$

For example, for two qubits, this initial set of generators is equivalent to

$$\{-\mathbf{e}_{13}, -(\mathbf{e}_{15} + \mathbf{e}_{24})/2, \mathbf{e}_{23}, (\mathbf{e}_{14} - \mathbf{e}_{25})/2\}. \tag{14}$$

The complete algebra is then found from all the new possible independent commutators calculated recursively until the linear space is exhausted [22]. From the complete algebra, only the Cartan subalgebra with $N/2$ elements can be used to define the field-free Hamiltonian of the system. As a result, the field-free Hamiltonian cannot have an arbitrary distribution of energy levels but must have the energy levels symmetrically distributed around an average (the offset) energy.

V. SUMMARY

A Lie algebra of $N(N+1)/2$ elements—significantly fewer than N^2 —is shown to be sufficient for arbitrary control of an even-level quantum system with symmetric energy levels, specifically of $n = log_2(N)$-qubit systems. All the elements of the algebra can be generated from a minimal set of N elements, which is the minimum number of generators for state control of the N-level system. These results have the potential to lead to more efficient optimal-control schemes for quantum state engineering and production of entangled states.

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